Non-Identity Measurement Models for Orientation Estimation Based on Directional Statistics

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***Abstract*—We propose a novel measurement update procedure for orientation estimation algorithms that are based on directional statistics. This involves consideration of two scenarios, orientation estimation in the 2D plane and orientation estimation in three-dimensional space. We make use of the von Mises distribution and the Bingham distribution in these scenarios. In the derivation, we discuss directional counterparts to the extended Kalman filter and a statistical-linearization-based filter. The newly proposed algorithm makes use of deterministic sampling and can be thought of as a directional variant of the measurement update that is used in well-known sample-based algorithms such as the unscented Kalman filter.**

Keywords*—*Bingham distribution, von Mises distribution, deterministic sampling, stochastic filtering, directional statistics.

I. Introduction

This paper contributes to the development of stochastic filtering algorithms for estimation of orientation from noisy measurements. The nature and uncertainty levels of these measurements may differ depending on the application and the functionality and quality of the underlying measurement devices, among which may be inertial measurement units (that involve a gyroscope, an accelerometer, and a magnetometer) or cameras. One of the special aspects that needs to be considered for orientation estimation is the fact that the domain of orientations is nonlinear and involves periodicity. For the planar case, the domain of orientations can be represented by the unit circle. Orientations in 3D-space may be represented as points on the 4-dimensional unit sphere.

Estimation techniques that assume a linear state space can only be used if the underlying uncertainties are small enough because adaptation of such techniques to the orientation estimation problem make use of local linearity of the underlying domains. Some of these approaches are discussed in [1], [2], [3], [4]. Handling of scenarios that involve high noise level requires consideration of the geometry of the underlying domain within the probability distribution. A simple approach to address this scenario would be the use of grid-based filters. However, these filters suffer from the curse of dimensionality and may require very dense (or at least adaptive) grids when the algorithm needs to be capable of handling changing noise levels (e.g., in order to account for changing sensing modalities). This results in potentially computationally burdensome filtering techniques that might be not suitable for practical applications.

Filtering approaches that are capable of handling different noise levels and consider the geometric structure of the underlying domain are typically based on directional statistics [5], [6], which is a subfield of statistics that investigates uncertain quantities defined on nonlinear manifolds (typically, the circle or the hypersphere). Filters that are based on directional statistics and applicable to orientation estimation were proposed in [7], [8], [9], [10]. They consider different types of system models and different underlying domains. However, all of them have in common that they assume a simple direct measurement model. This model is at most capable of considering the fact that the noisy measurement may be displaced by a fixed shift.

In linear state spaces, nonlinearities of system and measurement models can be addressed by a number of different techniques. Linearization is used within the extended Kalman filter (EKF) [11], which linearizes the system and measurement function around the current estimate. The statistical linearization filter [12] (that is also known as quasi-linear filter [13]) choses a linearization method that takes the uncertainty of the system state into account. Finally, implicit linearization is used in sample-based linear regression Kalman filters (LRKFs) [14], such as the unscented Kalman filter (UKF) [15] or the smart sampling Kalman filter (S^2KF) [16].

The main contribution of this paper is adapting some ideas of the measurement update within these filters to directional-statistics-based orientation estimation algorithms. Thus, a novel approximate measurement update procedure is proposed that is capable of handling models that are not merely assuming noisy direct measurements or a shifted variant of these. The algorithm is designed for a more general class of measurement functions that can be thought of as an orientation equivalent to nonlinear measurement models with additive noise in linear state-spaces. Consideration of these functions is achieved by explicitly approximating them by a simpler shift-based measurement model that gives rise to a closed-form update procedure. For estimating planar orientations, a procedure considering similar measurement models was proposed in [17]. It is based on a progressive measurement update scheme as presented in [18]. For the case of orientations in 3D space, this work presents the first approach that makes handling of this type of measurement models possible. The proposed procedure has a conceptual similarity to deterministic sampling-based nonlinear Kalman filters, such as the UKF or the S²KF. The resulting measurement update has a computational complexity that is comparable to the corresponding prediction step.

The remainder of this paper is structured as follows. In the next section, we revisit stochastic filtering algorithms that are based on the von Mises distribution (for planar orientation estimation) and the Bingham distribution (for estimating orientations in the three-dimensional case). The newly proposed measurement update algorithm is derived in Sec. III and its relation to comparable algorithms in linear state-spaces is discussed. An evaluation of the newly proposed algorithm is presented in Sec. IV where it is compared to a state-of-the-art approach. A discussion of the contribution and an outlook to future work is given in Sec. V.

II. Stochastic Filtering for Orientation Estimation

In the filtering approaches considered in this work, it is assumed that the orientation evolves according to the model

$$\underline{x}_{t+1} = a(\underline{x}_t) \oplus \underline{w}_t \ , \tag{1}$$

where $\underline{x}_t$ denotes the system state, $\underline{w}_t$ denotes the system noise, $a(\cdot)$ denotes the transition function, and $\oplus$ is a suitable shift operation, i.e., a group operation that accounts for rotations. Orientations in the plane can be represented by points on the unit circle $\mathbb{S}^1$ whereas orientations in 3D space can be represented as points on the 4D unit hypersphere $\mathbb{S}^3$ by using unit quaternions [19]. Furthermore, $\mathbb{S}^1$, $\mathbb{S}^3$ are the only hyperspheres that admit a topological group structure [20]. This is used for the definition of $\oplus$, and thus, the considered system model is not generalizable to other dimensions. The general course of action within the filtering algorithms discussed here is assuming the system state $\underline{x}_t$ and the noise $\underline{w}_t$ to be described by a certain family of distributions. It is not guaranteed that the transformation $a(x_t)$ or the operation $\oplus$ preserve this distribution family. Therefore, an approximate method is used to compute the predicted state $\underline{x}_{t+1}^p$ from the updated estimate $\underline{x}_t^e$.

So far, most directional filtering algorithms (e.g., [10], [7], [21]) assume noisy direct measurements. That is, the measurement model is given by

$$\underline{z}_t = \underline{x}_t \oplus \underline{v}_t \ , \tag{2}$$

where $\underline{v}_t$ represents the measurement noise. A suitable choice of the distribution families of $\underline{x}_t$ and $\underline{v}_t$ makes a closed-form measurement update possible. For a fixed $\underline{c}$, consideration of the more general case

$$\underline{z}_t = \underline{x}_t \oplus \underline{c} \oplus \underline{v}_t \tag{3}$$

is also possible by choosing a suitable (directional) mean of the noise term $\underline{v}_t$.

A. Orientation Estimation in the Plane

For the planar case, we will parametrize the unit circle $\mathbb{S}^1$ as the set $[0, 2\pi)$, and thus, the system state can be represented by a scalar value. The operator $\oplus$ is used to represent rotations in the plane. It is defined as

$$\oplus : \ \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 : \ (a, b) \mapsto (a + b) \mod 2\pi \ ,$$

which gives rise to an Abelian group structure.

In this work, the von Mises distribution is used for representing uncertain quantities on the circle [22]. Its density is given by

$$f(x) = \frac{1}{2\pi \, I_0(\kappa)} \exp(\kappa \, \cos(x - \mu)) \ ,$$

where I_0 denotes the modified Bessel function (see [23, Sec. 10.25]) of order 0, $\kappa \in \mathbb{R}^+$ is a concentration parameter and $\mu \in [0, 2\pi)$ is the location parameter. The notation $x \sim \mathrm{VM}(\mu, \kappa)$ is used to indicate that a random variable x is distributed according to this density. The product of two von Mises densities is again a (rescaled) von Mises density, which is a useful property that makes a closed-form measurement update possible.

Furthermore, deterministic sampling schemes are used for approximating the von Mises distribution by a discrete distribution. They are based on matching trigonometric moments, which are defined by

$$m_k = \mathbb{E}(e^{ikx}) \ .$$

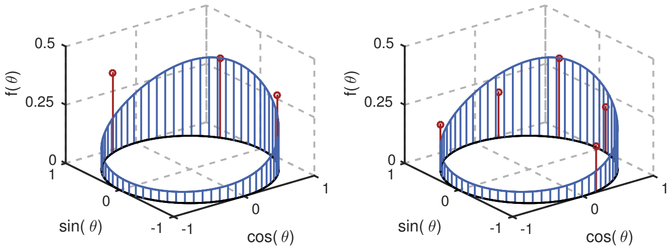
An approach that uses three samples to match the first trigonometric moment was proposed in [9]. It is directly applicable to approximating the von Mises distribution even though it was originally developed for approximating the wrapped normal distribution [24]. Later, an approach that uses five samples in order to match the first two circular moments was proposed in [25]. Both sampling schemes can be thought of as a circular counterpart to the deterministic sampling scheme within the UKF. An example, together with a corresponding von Mises distribution, is shown in Fig. 1.

The current estimate before prediction is given by a $\mathrm{VM}(\mu_t^e, \kappa_t^e)$ distribution and the noise term is assumed to follow a $\mathrm{VM}(\mu^w, \kappa^w)$ distribution. The prediction step is based on using deterministic sampling in order to approximately compute the first trigonometric moment and then obtaining the von Mises distribution parameters μ_{t+1}^p, κ_{t+1}^p. This requires the use of a numerical procedure because the first trigonometric moment of a $\mathrm{VM}(\mu, \kappa)$ distribution is given by

$$m_1 = \frac{I_1(\kappa)}{I_0(\kappa)} \mathrm{e}^{i\mu} \ .$$

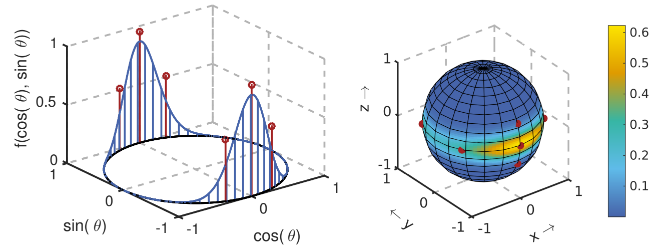
For the wrapped normal case, the entire prediction procedure is discussed in more detail in [9].

Use of the von Mises distribution within a measurement update was originally proposed in [7]. This makes a closed-form measurement update possible when the observation model is given by (2). The entire update step is then carried out as follows. After obtaining a new measurement z, prior parameters



(a) 3 Samples [9]. (b) 5 Samples [25].

Figure 1: Example for deterministic Sampling of the von Mises distribution.



(a) 2D case. (b) 3D case.

Figure 2: Example for deterministic sampling of the Bingham distribution.

μ_t^p, κ_t^p and noise parameters μ^v, κ^v are used to obtain the new estimate according to [7] as

$$\begin{aligned} \tilde{\mu}^v &:= (z_t - \mu^v) \ , \\ C &:= \kappa_t^p \cos(\mu_t^p) + \kappa^v \cos(\tilde{\mu}^v) \ , \\ S &:= \kappa_t^p \sin(\mu_t^p) + \kappa^v \sin(\tilde{\mu}^v) \ , \\ \mu_t^e &:= \operatorname{atan2}(S, C) \ , \\ \kappa_t^e &:= \sqrt{C^2 + S^2} \ . \end{aligned}$$

B. Orientation Estimation in 3D Space

The entire picture looks similar for the three-dimensional case. The hypersphere $\mathbb{S}^3$ is represented as unit vectors in $\mathbb{R}^4$, i.e., the set $\{\underline{x} \in \mathbb{R}^4 \,|\, ||\underline{x}|| = 1\}$ which corresponds to unit quaternions. The corresponding shift operation $\oplus$ is given by quaternion multiplication that is also known as the Hamilton product. It is defined [19, Sec. 3] as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \oplus \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} := \begin{pmatrix} a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 \\ a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 \\ a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 \\ a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 \end{pmatrix} \ .$$

Furthermore, a probability distribution is required that accounts for the fact that the quaternions $\underline{q}$ and $-\underline{q}$ represent the same orientation. This is achieved by using the Bingham distribution [26] which is antipodally symmetric and defined on arbitrary-dimensional hyperspheres. Its p.d.f. is given by

$$f(\underline{x}) = \frac{1}{N(\mathbf{Z})} \exp(\underline{x}^\top \mathbf{M} \mathbf{Z} \mathbf{M}^\top \underline{x}) \ , \qquad ||\underline{x}|| = 1 \ ,$$

where $||\underline{x}|| = 1$, $N(\mathbf{Z})$ is the normalization constant, $\mathbf{M}$ is an orthonormal matrix that can be thought of as a location parameter (because it describes the position of the modes and the location of the axes), and $\mathbf{Z}$ is a diagonal matrix that can be thought of as a concentration parameter. A filter that is based on the Bingham distribution and assumes an identity system model was proposed in [10], [21], [27].

A sampling scheme of the Bingham distribution was proposed in [28]. It is based on matching the second moment $\mathbb{E}(\underline{x}\,\underline{x}^\top)$ of a Bingham distributed random vector $\underline{x}$. This is achieved by generating a set of $4n - 2$ samples to approximate a Bingham distribution in $\mathbb{R}^n$. There is no intuitive visualization of the Bingham distribution for the 4-dimensional case. Therefore, the sampling scheme was visualized for the two-dimensional and three-dimensional cases together with the corresponding Bingham densities in Fig. 2.

Once again, the sampling scheme can be used to handle the system model (1). The general course of action is similar to the circular filters. That is, the system state is represented by a $\mathrm{Bingham}(\mathbf{M}_t^e, \mathbf{Z}_t^e)$ distribution. Analogously, the noise is assumed to be distributed according to $\mathrm{Bingham}(\mathbf{M}^w, \mathbf{Z}^w)$. The predicted $\mathrm{Bingham}(\mathbf{M}_{t+1}^p, \mathbf{Z}_{t+1}^p)$ is obtained by deterministic sampling, subsequent propagation, and matching of the second moment. This entire procedure is described in [28].

The measurement update step, when the observation model (2) is assumed, is also similar to the von Mises-based approach. After obtaining measurement $\underline{z}$, the parameters $\mathbf{M}_t^p$, $\mathbf{Z}_t^p$ of the prior and the parameters $\mathbf{M}^v$, $\mathbf{Z}^v$ of the measurement noise are used to obtain the parameters of the posterior from the eigendecomposition of

$$(\underline{z}_t \oplus \mathbf{M}^v)\, \mathbf{Z}^v \,(\underline{z}_t \oplus \mathbf{M}^v)^\top + \mathbf{M}_t^p \, \mathbf{Z}_t^p \,(\mathbf{M}_t^p)^\top \ ,$$

where $(\underline{z}_t \oplus \mathbf{M}^v)$ denotes quaternion multiplication of $\underline{z}_t$ with each column of $\mathbf{M}^v$. This yields eigenvectors α_i and eigenvalues $\underline{e}_i$. This yields $\mathbf{Z}_t^e = \mathrm{diag}(\alpha_1, ..., \alpha_4)$ and $\mathbf{M}_t^e = (\underline{e}_1, ..., \underline{e}_4)$.

III. New Measurement Update for Orientation Estimation

This chapter aims to overcome the limitations of the measurement model that is used in the filters presented in the previous section. Our goal is to develop approximate techniques for handling the model

$$\underline{z}_t = h(\underline{x}_t) \oplus \underline{v}_t \ , \tag{4}$$

where we assume $\underline{z}_t$ and $\underline{v}_t$ to be defined on the same domain as the system state. The development of the proposed algorithm can be subdivided in three steps. First, this measurement model is approximated by a fixed shift $\underline{c}$ that depends on the current estimate. In the second step, this shift is chosen under consideration of the true uncertainty of the system state. This can be thought of as a statistical shift. Finally, this approximation is refined by updating the parameters of the noise term in order to account for the error that is made by implicitly assuming the measurement model to be (2). All of these approaches are motivated by linear counterparts where a complicated nonlinear function is (either explicitly or implicitly) approximated by a linear function which makes a closed-form measurement update possible. The following presentation will be given in general terms. That is, it is applicable to both cases, orientation estimation in the plane and in 3D space.

The notation $\hat{\underline{x}}_t^p$ and $\hat{\underline{x}}_t^e$ will be used for representing the predicted system state and its updated estimate. In the planar case this is given by the angular mean μ_t^p and μ_t^e of the von Mises distributions involved. For the Bingham case, this is given by the eigenvector of $\mathbf{M}_t^p \mathbf{Z}_t^p (\mathbf{M}_t^p)^\top$ that corresponds to the largest eigenvalue (and analogously for the updated estimate). This does not require yet another numerical procedure, because the location parameter $\mathbf{M}$ of a Bingham distribution is an orthonormal matrix, and thus, the mode is obtained as the column of $\mathbf{M}$ that corresponds to the largest value of $\mathbf{Z}$. The densities that correspond to the estimates and the noise terms will be denoted by f_t^p, f_t^e, f^w, and f^v.

In the following algorithms, $\ominus$ will be used to denote the application of an inverse shift. For the circular case, this is given by

$$a \ominus b = a \oplus (-b) = (a-b) \mod 2\pi \ .$$

It is important to note that inversion with respect to both group structures presented above preserves the distribution family. That is, the (multiplicative) quaternion inverse of a Bingham distributed random vector is itself a Bingham distributed random vector. This holds analogously for the von Mises case.

A. Approximation by a Shift

Approximating the measurement model by a shift is simply carried out by reformulating

$$\underline{z}_t = h(\underline{x}_t) \oplus \underline{v}_t \approx \underline{x}_t \oplus \underline{c} \oplus \underline{v}_t \ .$$

After computing $\underline{c}$, the measurement update can be performed as presented in the preceding section. A first naïve approach of obtaining $\underline{c}$ is based on adapting the idea of the extended Kalman filter (EKF) [11] to the directional setting. That is, $\underline{c}$ is chosen according to

$$\underline{c} = h(x_{t+1}^p) \ominus x_{t+1}^p \ .$$

The entire resulting filtering algorithm is visualized in Algorithm 1. There (and in the following algorithms), the function $\textsc{UpdateIdentity}(f_t^p, \tilde{f}^v, \underline{z})$ is used to denote the identity measurement update step that assumes (2).

Algorithm 1 Shift-based Approximation

```
procedure MEASUREMENTUPDATE(x̂_t^p, f^v, z, h(·))
    c ← h(x̂_t^p) ⊖ x̂_t^p;
    f_t^e ← UPDATEIDENTITY(f_t^p, f^v, z ⊖ c);
    return f_t^e
end procedure
```

So far, the algorithm is expected to yield good performance whenever the measurement model resembles a shift based model locally. Due to the relationship with the EKF, this approach also suffers from similar drawbacks. That is, the linearized approximation of the true function of the EKF purely depends on the current point-estimate. Similarly, $\underline{c}$ depends on $\hat{\underline{x}}_p^t$ and not on the uncertainty of this estimate. Thus, the approximation may yield poor results in cases of higher noise or strong nonlinearities.

B. Approximation by a Statistical Shift

In this step, the goal is to improve the quality of the estimate by choosing $\underline{c}$ in a better way. The idea is to develop a directional analogue to the statistical linearization filter [12]. For the circular case, this is done by matching the first trigonometric moment. That is, finding a $c \in [0, 2\pi)$ such that

$$\mathbb{E}(\exp(i(h(x_t) \ominus x_t))) = \exp(i\, c) \ .$$

For the quaternion case, an analogous procedure is carried out that matches the second moment (which in that case is identical with the covariance matrix). Computing this expectation numerically in each filter step might be burdensome. Thus, we use a sample-based approach that approximates the expected values involved.

Algorithm 2 Statistical Shift

```
procedure MEASUREMENTUPDATE(f_t^p, f^v, z, h(·))
    (s_{x,i}, p_{x,i})_{i=1,...,L} ← DETSAMPLING(f_t^p);
    for all i ∈ {1, ..., L} do
        s_{d,i} ← h(s_{x,i}) ⊖ s_{x,i};
    end for
    c ← GETDIRECTIONALMEAN((s_{d,i}, p_{x,i})_i);
    f_t^e ← UPDATEIDENTITY(f_t^p, f^v, z ⊖ c);
    return f_t^e
end procedure
```

The entire resulting algorithm is visualized in Algorithm 2. There, the procedure $\textsc{GetDirectionalMean}((\underline{s}_{d,i}, p_{x,i})_i)$ denotes a dimension-dependent mean computation. For the circular case, we make use of [6, Sec. 1.3.1] and obtain c as

$$\begin{aligned} c = \operatorname{atan2}\Big(&\sum_{i=1}^{L} p_{d,i} \sin(h(s_{d,i}) - s_{d,i}), \\ &\sum_{i=1}^{L} p_{d,i} \cos(h(s_{d,i}) - s_{d,i}) \Big) \ . \end{aligned}$$

For the quaternion case, the mean orientation $\underline{c}$ is obtained as the eigenvector corresponding to the largest eigenvalue of

$$\sum_{i=1}^{L} p_{x,i} \cdot \underline{s}_{d,i} \cdot \underline{s}_{d,i}^\top \ ,$$

which corresponds to the mode computation of the Bingham distribution.

The statistical shift is better at capturing local behavior of the true measurement function for the choice of $\underline{c}$. Thus, it promises to yield better results. However, in this approach the measurement update itself does not account for the fact that the measurement function $h(\cdot)$ might impact the uncertainty of $\underline{x}_t$. That is, the noise parameters of $\underline{v}_t$ remain unchanged.

C. Correction of Noise Term

So far, $\underline{v}_t$ was used to represent the measurement noise. In this final step, our goal is to update the parameters of the noise term in order to account for the additional uncertainty that stems from the fact that the approximate measurement model differs from the real measurement model. Our approach was inspired by the LRKF in which the additional uncertainty that stems from linearization errors is accounted for by the covariance of the deviations between the nonlinear function and its implicitly linearized counterpart (see [14, Sec. IV]). This idea is adapted to the directional case by approximating the entire measurement model (4) as follows

$$\underline{z}_t = h(\underline{x}_t) \oplus \underline{v} \approx \underline{x}_t \oplus \underline{\tilde{v}} \ ,$$

where $\underline{\tilde{v}}$ belongs to the same distribution family as $\underline{v}$ and has different noise parameters, which can also account for a shift within the location parameter. Thus, this approximation generalizes the previous approach. Once again, deterministic sampling is used in order to estimate the parameters of $\underline{\tilde{v}}$.

The measurement update is carried out as follows. Deterministic sampling of the predicted state density f_t^p and the noise density f^v is used in order to obtain a sample-based approximation of $h(\underline{x}_t) \oplus \underline{v}_t \ominus \underline{x}_t$. Then, these samples are used in a moment-matching based density estimation procedure to obtain the density $\tilde{f}^v$ of $\underline{\tilde{v}}$. Finally, once again the measurement update is performed under the assumption of (2) as the measurement model.

The entire resulting procedure is visualized in Algorithm 3. There, the function EstimateDensity$((\underline{s}_{d,i}, p_{d,i})_{i=1,\ldots,L\cdot M})$ performs density estimation from weighted samples which is discussed in more detail in papers on sample based directional filtering, i.e., in [9] and [28].

Algorithm 3 Noise correction.

```
procedure MeasurementUpdate(f_t^p, f^v, ẑ, h(·))
    (s_{x,i}, p_{x,i})_{i=1,...,L} ← DetSampling(f_t^p);
    (s_{v,i}, p_{v,i})_{i=1,...,M} ← DetSampling(f^v);
    k ← 1;
    for all i ∈ {1, ..., L} do
        for all j ∈ {1, ..., M} do
            s_{d,k} ← h(s_{x,i}) ⊕ s_{v,j} ⊖ s_{x,i};
            p_{d,k} ← p_{x,i} · p_{v,j};
            k ← k + 1;
        end for
    end for
    f̃_t^v ← EstimateDensity((s_{d,i}, p_{d,i})_{i=1,...,L·M});
    f_t^e ← UpdateIdentity(f_t^p, f̃_t^v, z);
    return f_t^e
end procedure
```

Once again, this algorithm promises better results compared to the approach that uses an intelligent way to compute the shift term $\underline{c}$. However, it still involves the implicit approximation of the true noise model by a simpler model, and thus, may significantly differ from the possibly computationally intractable Bayesian estimator that does not involve any approximation.

IV. Evaluation

The evaluation is carried out by performing measurement update steps rather than by simulating the entire filter run. This avoids the introduction of approximation errors within the prediction step, and thus, gives a better picture of the properties of the newly proposed algorithm. Furthermore, for better visualization and more intuitive interpretation, all simulations were performed for the planar orientation estimation case, i.e., the considered

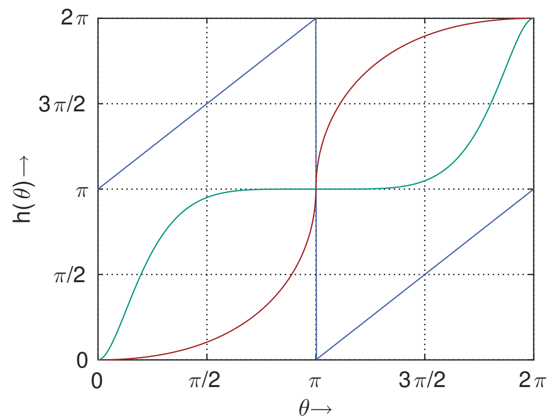


Figure 3: Measurement model $h_a(\theta)$ for $a = 0.5$ (red), $a = 5$ (green), and the shifted model (blue).

state space is $\mathbb{S}^1$. This is justified by the fact that the quaternion-based filtering algorithm has a similar structure, and thus, the big picture is expected to look similar.

The comparison uses two types of measurement models. First, we consider a somewhat complicated function that is given by

$$h_a(\theta) = \pi \cdot \left(\sin\left(\frac{\operatorname{sign}(\theta - \pi)}{2} \cdot \frac{|\theta - \pi|^a}{\pi^{a-1}} \right) + 1 \right)$$

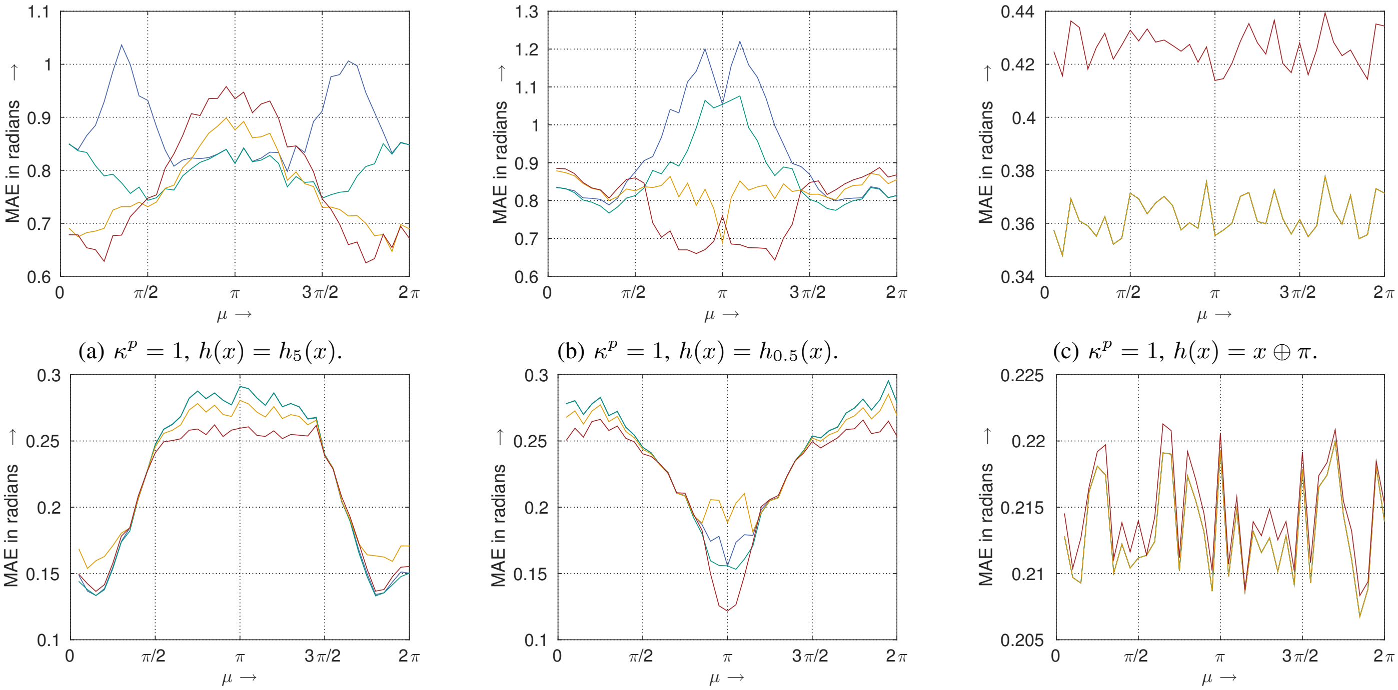
for $\theta \in [0, 2\pi)$, $a \in \mathbb{R}_+$. Here, a serves as a shape parameter. Second, we consider shifted noisy measurements. In particular, we use $h(x_t) = x_t \oplus \pi$ which corresponds to (2) with a suitably chosen mean of the noise term $\underline{v}_t$. All considered measurement models are visualized in Fig. 3.

The simulation is carried out as follows. First, we generate a sample x that represents the system state by sampling randomly from $\mathrm{VM}(\mu, \kappa^p)$. Then, this sample is propagated through the considered measurement function and a noise sample v is generated from $\mathrm{VM}(0, \kappa^v)$. Then, a measurement is obtained according to $z = h(x) \oplus v$. This measurement is used to perform an update step using different the algorithms discussed above and additionally the progressive algorithm from [17]. In this update step the prior estimate is given by the $\mathrm{VM}(\mu, \kappa^p)$ distribution. Finally, the error is computed as the angle between the true state and the updated estimate, i.e., the error measure is given by

$$e(\theta_1, \theta_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|) \ .$$

The simulation itself is carried out for different values of μ. For the concentration of the measurement noise, we assumed $\kappa^v = 5$. Two cases were considered for the concentration of the prior, $\kappa^p = 1$ and $\kappa^p = 10$. The angular error measure was averaged over 2000 runs which results in a mean angular error (MAE) that is used as a quality measure. The results are visualized in Fig. 4.

The results do not show one single approach to be superior in all cases, which is not surprising because the considered scenarios involve all types of extreme cases that may give certain approaches an advantage. The general picture, however, is that the progressive update algorithm yields the best results as



(a) $\kappa^p = 1$, $h(x) = h_5(x)$. (b) $\kappa^p = 1$, $h(x) = h_{0.5}(x)$. (c) $\kappa^p = 1$, $h(x) = x \oplus \pi$.

(d) $\kappa^p = 10$, $h(x) = h_5(x)$. (e) $\kappa^p = 10$, $h(x) = h_{0.5}(x)$. (f) $\kappa^p = 10$, $h(x) = x \oplus \pi$.

Figure 4: Results of the simulation run using the approaches based on a shift approximation (blue), a statistical shift (green), the noise correction approach (orange), and the progressive approach from [17] (red).

long as the measurement is close to the true system state. It may become very poor when the measurement is far away which is seen in the example in which shifted direct measurements are used. There, all three algorithms that were discussed in this paper yield an equivalently correct result. Furthermore, the progressive algorithm is the most expensive from a computational viewpoint due to repeated reapproximations. The noise correcting algorithm usually outperforms the approaches based on a shift or a statistical shift, but it also suffers from a higher computational demand than these two naïve approaches.

V. Discussion and Outlook

This work proposed a new algorithm for the measurement update in orientation estimation algorithms that are based on directional statistics. The key idea of the newly proposed algorithms was adapting certain concepts that are well-known for the case of estimating linear quantities to the directional setting. In particular, it can be thought of as a directional equivalent of filters based on statistical linearization. Thus, the advantages and drawbacks of the newly proposed approach are similar to its linear counterparts. On the one hand, it suffers from implicitly assuming the measurement model to be an identity model which can at most account for a shift. On the other hand, it benefits from avoiding problems of other sample-based approaches, such as, particle degeneration or possibly costly repeated resampling within one single measurement update step. Thus, the newly proposed approach offers a good tradeoff between accuracy and computational complexity.

There are several interesting directions for possible future work. First, it is of some interest to investigate measurement models that assume the measurement noise or even the measurement itself to be defined on an entirely different domain as the system state. Second, it is of interest to investigate probability distributions that are capable of representing multiple uncertain orientations and their respective dependencies for the development of more general filtering approaches. Finally, it is of some interest to adapt the ideas presented in this work for the development of a filter that considers position and orientation simultaneously and involves possibly complicated measurement models.

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